Number in the senior primary classes

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Importance and Relevance of Number

Number connects all primary mathematics curriculum strands. As number knowledge supports students' performance in algebra, measures, geometry, and data and probability, these strands support learning number.

Predictor of Success

Early number competence predicts later success in mathematics generally (Geary et al, 2013; Jordan et al, 2010; Nguyen et al, 2016; Shanley, 2016). A challenging curriculum should develop students' number sense in upper primary classes (Cason et al., 2019). Students' knowledge of fractions and division of whole numbers can better predict the likelihood of success in post-primary school mathematics than variables like knowing whole number operations other than division, verbal and non-verbal IQ, working memory, family education, and family income (Siegler et al, 2012).

Prominent in Tests

Number questions have traditionally featured strongly in national assessments; 69 questions (46%) in 2014 were on number/algebra. Irish students answered 63% of number/algebra questions correctly, higher than on measures but similar to data and geometry (Shiel et al, 2014). In 2015 Irish fourth class students achieved an average scale score (centred on 500, SD 100) of 551 on TIMSS number items, indicating a strength in number compared to the overall average scale score of 547 and a score of 542 for geometric shapes and measures (Clerkin et al, 2016). In the TIMSS 2019 Mathematics Framework, 50% of fourth class items were devoted to number, including pre-algebra (Lindquist et al, 2019).

Connected to algebra

In upper primary school number incorporates further consolidation of place value and base ten, operations on whole numbers and fractions, and rational numbers (fractions, decimals and percentages). Early number work provides the foundation for subsequent development of algebraic understanding; students learn about properties of number operations (commutative, associative, distributive, zero, identity) and conceive the equals sign as representing a relation between equal quantities rather than an indication to calculate (Carpenter et al, 2003). In problems like 4 + 5 = __ + 3, where incorrect answers 9 or 12 are frequently given throughout primary school, misunderstanding of the equals sign is evident (Carpenter et al, 2003).

Number Sense

Little consensus exists on what constitutes number sense (Reys & Yang, 1998). It likely includes understanding meanings of numbers and operations, recognising number size, representing numbers and operations in multiple ways, recognising the effects of operations on numbers, and evaluating the reasonableness of calculation results. It includes place value; meanings and properties of the four operations; comparing and ordering numbers; using pictorial and symbolic representations; solving number problems flexibly; mental computation and estimating (Yang, 2019). Asking students to indicate confidence in their answers to questions may indicate how they perceive their number competence (Siegler et al, 2011; Yang, 2019).

Curriculum, Number and Hypothetical Learning Trajectories

Hypothetical learning trajectories (Simon, 1995) may help develop progression continua for the curriculum. A learning trajectory is hypothesised based on two pillars linked to one common mathematical learning goal:

- a model of how children learn in a domain moving from concrete to abstract and naïve to sophisticated and
- a sequence of tasks that helps students construct concepts to advance their learning in the domain (Clements & Sarama, 2004).

Hypothetical Learning trajectories have been proposed for number topics including equipartitioning, equal composition or decomposition (Baroody et al, 2004; Confrey & Maloney, 2015), the successor principle in counting (Baroody et al, 2019), integer addition and subtraction (Stephan & Akyuz, 2012), relational thinking (Stephens & Armanto, 2010), and a meaning of multiplication that subsumes both whole numbers and fractions (Simon et al, 2018).

Learning trajectories are valuable for teachers at school or class level (Clements and Sarama, 2004) and even for individuals' mathematics learning (Steffe, 2004). Although some criticise hypothetical learning trajectories as being too detailed or prescriptive to help practising teachers, using them alongside big mathematical ideas may optimise their potential (Baroody et al, 2004).

Acquiring Number Concepts

Vergnaud (2009) describes a concept as having a set of:

- situations,
- operational invariants, and
- linguistic and symbolic representations.

In order for children to develop a conceptual field – say, for multiplication – they must encounter contrasting situations (Vergnaud, 2009). Typically in curricula and textbooks, topics such as multiplication, division, fractions, ratio and proportion, and rational numbers are presented and taught separately despite being closely interrelated (Vergnaud, 1983; Lampert, 2001).

Choosing and using examples and non-examples carefully help students acquire concepts. For example, 2 is a weak example of a prime number but as one of several examples, it is useful. Students should participate in constructing examples (Goldenberg & Mason, 2008).

Numbers in Primary School

Primary school students encounter natural numbers, whole numbers, integers, and rational numbers (see Figure 1). Pi (π) is an irrational number which occasionally arises when investigating the relationship between a circle's circumference and diameter.

Properties and rules of operations change from one set of numbers to another. In first grade (first class) one cannot subtract 8 from 5 using whole numbers; however, that is no longer true in fifth grade (fifth class) when integers are introduced. One common misconception is that multiplying a number makes it bigger and dividing a number makes it smaller. Although this applies to natural numbers (except when multiplying or dividing by 1), it is untrue for whole numbers (e.g. multiplying by 0), integers (e.g. multiplying by -1) and rational numbers (multiplying or dividing by ½ or 0.5) (Graeber & Tirosh, 1990). The curriculum can specify the intended domain of numbers in learning outcomes or progression continua.

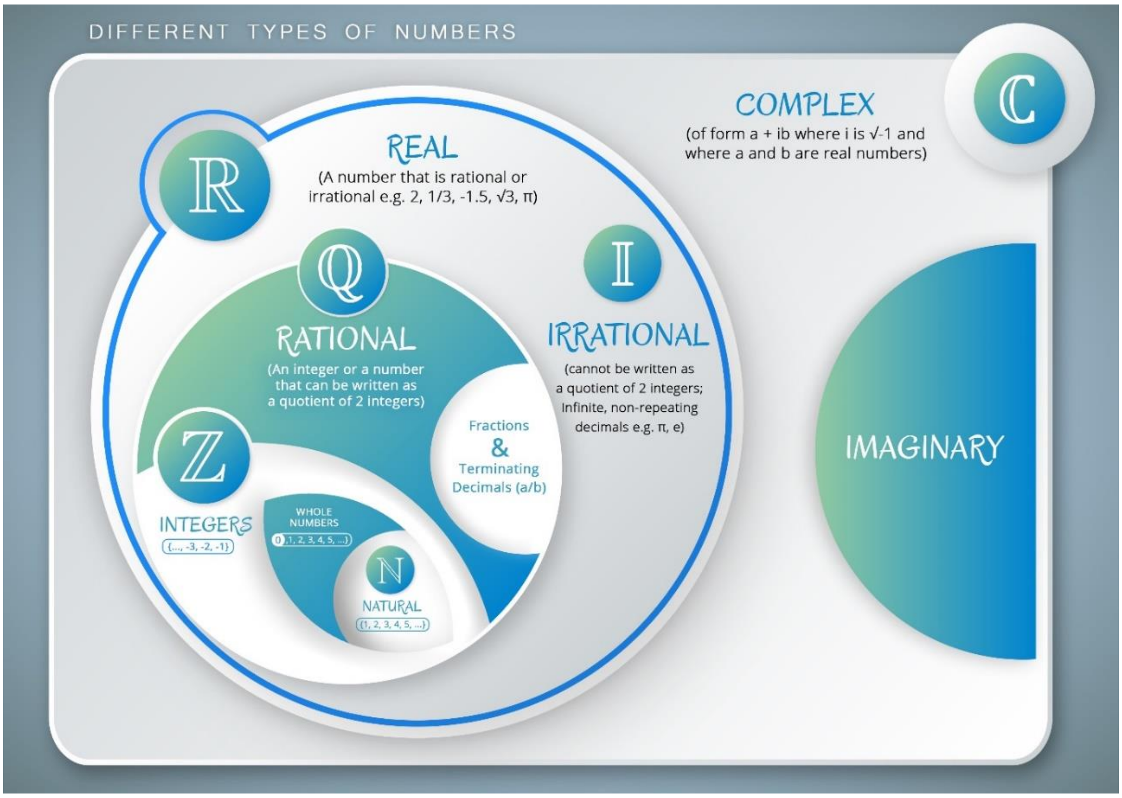


Figure 1. Different types of numbers.

Key Concepts

Place Value

The powerful concept of place value is currently introduced formally in first class (Government of Ireland, 1999). Throughout the remaining primary school years students expand their place-value knowledge in order to understand that the value of a number such as 123.45 is derived as follows:

$$123.45 = 100 + 20 + 3 + 4/10 + 5/100$$

$$= (1x100) + (2x10) + (3x1) + (4x1/10) + (5 \times 1/100)$$

$$=1 \times (10x10) + 2 \times 10 + 3 \times 1 + 4 \times {}^{1}/_{10} + 5 \times ({}^{1}/_{10} \times {}^{1}/_{10})$$

$$= 1 \times 10^{2} + 2 \times 10^{1} + 3 \times 10^{0} + 4 \times 10^{-1} + 5 \times 10^{-2}$$

Knowing each digit's value requires knowledge of multiplication. Knowing the total value of the number requires knowledge of addition.

The base ten place value system represents number using the structure of algebra. Because it is ubiquitous, it may be considered trivial. However, understanding the complexity and power of place value requires drawing on mathematics encountered during students' eight years in primary school, from addition to multiplication to exponents. Therefore, teaching place value continues throughout primary school (Howe, 2019).

Concrete Materials and Symbol-Based Instruction

Extensive exposure to relevant concrete materials benefits students' learning about place value, especially its structure, even if such benefits only become apparent over time. Learners, particularly higher-achievers, also benefit from symbol-based instruction, especially in relation to estimating on a number line. However, further research is needed into the relative benefits of approaches using concrete materials and using symbols, which appear to be context-dependent (Mix et al, 2017). Students' understanding of place value in first class is correlated with students' grasp of decimal and fraction magnitude in fourth class, confirming the importance of establishing a strong foundational understanding of place value from the early years (Wong, 2019).

Numeration Units

To support place value understanding, numeration units can be used to represent numbers (e.g. 741 as 7 hundreds, 4 tens and 1 unit) alongside the usual written and spoken forms. However, further research is required to confirm the benefits of this approach for student learning (Houdement & Tempier, 2019).

Decimal Numbers

When naming decimal numbers (e.g. 0.24) teachers may use informal labels such as “point two four,” formal labels such as “two tenths and four hundredths,” or dominant place value labels such as “twenty four hundredths.” Using the formal labels has some advantages in understanding the relationship between tenths and hundredths and in comparing the magnitude of numbers involving zero in a decimal place. However, students who were taught to use formal labels performed worse when estimating on the number line (Loehr & Rittle-Johnson, 2017). Although further research in this area is required, using varied labels for decimals seems advisable because formal labels may draw attention to the place value structure of decimal numbers.

From Additive to Multiplicative Thinking

A major mathematical transition when moving from junior to senior primary school classes is from additive thinking to multiplicative thinking. Some students demonstrate features of multiplicative thinking in second class but typically it develops slowly; one US study found that only about half of fifth-grade children consistently demonstrated such thinking (Clark & Kamii, 1996).

When choosing a problem-solving strategy, students rely on superficial cues like how a problem is formulated, the numbers used, key words or expressions, or the title of the chapter the problem appears in. Such strategies may lead students to use an incorrect operation (Van Dooren et al, 2010). Children’s tendency to use numbers in a problem to determine which strategy to apply suggests that numbers should be chosen carefully in student tasks and materials, such as examples, textbooks and assessments (Van Dooren et al, 2010). Proportional strategies are more likely to be used when numbers in the problem formed integer ratios; when the numbers did not form integer ratios, additive strategies were more likely to be used. Consider the following task.

Maya and Kai are making paper cranes. They start at the same time but Maya is faster than Kai. When Maya has made 8 cranes, Kai has made 6. When Maya has made 12 cranes, how many has Kai made?

This problem requires a multiplicative strategy, but many students use an additive approach because the relevant numbers constitute a non-integer ratio. Therefore instead of calculating this problem correctly as $^{6}/_{8} = {}^{\mathbf{9}}/_{12}$ (i.e. $^{b}/_{a} = {}^{x}/_{c}$), the problem is frequently calculated incorrectly as 12 – 8 = **10** - 6 (i.e. b-a=x-c), where a, b, and c are given and x must be figured out.

In third class, students are more likely to apply additive strategies to all problems – proportional or additive – whereas by fifth class the opposite applies and students more likely to apply multiplicative strategies to both kinds of problems. One widely-used approach that may hinder students' development of multiplicative thinking is introducing multiplication solely as repeated addition.

Difficulties with Introducing Multiplication Using Repeated Addition

Multiplication is frequently introduced in primary school as repeated addition. Although repeated addition makes calculation relatively easy in the short term (Delaney, 2017), it constrains students' thinking as additive rather than advancing them towards multiplicative reasoning (Schwartz, 1988). The calculation 6 x 5 can be conceived as 5 + 5 + 5 + 5 + 5 + 5 (repeated addition) or it can preferably be imagined multiplicatively as a number that is 6 times as large as 5 (Thompson & Saldanha, 2003).

Multiplicative understanding helps children see a product, say *mn*, as being in multiple reciprocal relationships with *m* and *n*: i.e. *nm* is *n* times as large as *m*; *mn* is *m* times as large as *n*; *m* is 1/*n* as large as *nm*, and *n* is 1/*m* times as large as *nm* (Thompson & Saldanha, 2003). Presenting multiplication as repeated addition makes it more difficult to apply multiplication to measurement, rational numbers, ratio, and proportion. For example, in $^2/_3$ x ¾ it is difficult to imagine adding ¾ to itself $^2/_3$ times.

Units Coordination

Repeated addition may hinder students' ability to coordinate units. Units coordination is the idea that, for example, the month of February in a non-leap year may be thought of as 28 individual days, 4 weeks of 7 days, or one month of 28 days (the unit changing from day, to week, to month). Initially a student can take one level as given and coordinate a second level; next a student may take two levels as given but can coordinate a third level; and at a third stage a student may simultaneously grasp all three levels and switch from one unit to another (Clark and Kamii, 1996; Norton et al, 2015). In fractions the levels may refer to the whole unit, a unit fraction, and then a non-unit fraction (Simon et al, 2018). Although students may develop the levels using materials or diagrams, they have successfully mastered a stage when they have interiorised one or more levels (Simon et al, 2018). Students may be competent in calculating products (if they have memorised tables, for example) but without opportunities to develop unit coordination, their ability to modify such calculations may be limited (Norton et al, 2015).

Second, introducing multiplication through repeated addition ignores the fact that whereas addition is referent preserving, multiplication is referent transforming (Schwartz, 1988; Simon & Placa, 2012; Vergnaud, 1983). If you have 6 apples and you add 3 apples, you have 9 apples; the unit, apples in this case, remains constant. However, if you have 6 kilos of apples and each kilo costs €3, then the cost is €18; the units attached to each number differ, from (a) kilos of apples, to (b) cost per kilo of apples to (c) cost in Euro. The cost per kilo is referred to as an intensive quantity (Schwartz, 1988) because it is a quality of the apples (applying to one apple or a box of apples) that cannot be counted or measured directly, unlike an extensive quantity (e.g. number of apples, length of a line etc.) that can be measured directly (Schwartz, 1988; Simon et al, 2012). Similarly, when calculating area, two measures of length (say metres) are multiplied and the product is square metres (Izsák & Beckmann, 2019).

Multiplication

The challenge of finding time to cover the curriculum can be addressed by making connections across mathematical content and processes (Lampert, 2001). Multiplication is an ideal topic for making cross-curriculum connections. In trying to capture the interconnectedness of multiplicative concepts, which students acquire over several years, Vergnaud (1983, 1988, 1994) proposed the idea of a multiplicative conceptual field. Conceptual fields classify mathematical relationships that help students move from using intuitive strategies to developing intuitive knowledge which is transformed into explicit knowledge (Vergnaud, 1988). Teachers can use conceptual fields to diagnose what students know and don't know, to identify situations that can be offered to consolidate and increase students' knowledge and recognise its limits at a particular time.

Situations that can be modelled by multiplication and division have been summarised with examples by Greer (1992, cited in Simon et al, 2018, p. 152):

1. *Equal groups*. 3 children each have 4 oranges. How many oranges do they have altogether?
2. *Equal measures*. 3 children each have 4.2 litres of orange juice. How much orange juice do they have altogether?
3. *Rate*. A car moves at a steady speed of 4.2 m per second. How far does it move in 3.3 s?
4. *Measure conversion*. An inch is about 2.54 centimetres. How long is 3.1 inches in centimetres?
5. *Multiplicative comparison*. Iron is 0.88 times as heavy as copper. If a piece of copper weighs 4.2 kg, how much does a piece of iron the same size weigh?
6. *Part/whole*. There are 30 students in a class, $^2/_5$ of the students are boys. How many boys are in the class?
7. *Multiplicative change*. A piece of elastic can be stretched to 3.3 times its original length. What is the length of a piece 4.2 m long when fully stretched?
8. *Cartesian product*. If there are 3 routes from A to B and 4 routes from B to C, how many different ways are there of going from A to C via B?
9. *Rectangular area*. What is the area of a rectangle 3.3 m long by 4.2 m wide?

10. *Product of measures*. If a heater uses 3.3 kilowatts of electricity for 4.2 h, how many kilowatt-hours were used? (p. 280)

The situations listed above illustrate the kind of tasks students can solve when developing multiplicative reasoning in senior primary school classes.

Researchers have attempted to unite meanings of multiplication in order to build on prior knowledge when constructing students' understanding of multiplication. One such attempt (Izsák & Beckmann, 2019) links multiplication and measurement and is based around three questions related to the multiplier, multiplicand and product, as follows:

N	*M*	*P*
"How many base units make one group exactly?"	"How many groups make the product amount exactly?"	"How many base units make the product amount exactly?"
Multiplier	Multiplicand	Product

Figure 2. A summary definition of multiplication based in coordinated measurement. Questions listed are direct quotations from Izsák and Beckmann (2019, p. 91).

Although such an approach holds promise for the future, this research is currently at an early stage and requires further refinement before it will inform curriculum design more generally.

Number triples are sets of three numbers where one number is the product of the other two (Downton & Sullivan, 2017). Examples include 3,4,12; 13,7,91. Triples can be used in multiplication tasks such as those listed above. Downton and Sullivan (2017) suggest that rather than presenting simpler, graded, number triples in tasks for students, engaging with more complex tasks, and giving students a choice in the level of challenge they experience, may elicit the use of more sophisticated strategies by students.

Difficulties students have with multiplication include

- adding instead of multiplying;
- calculating successfully without understanding, and consequently being unable to compose a relevant story for the calculation; and
- being unable to figure out a new product from a known one, say knowing 7 x 7 but not being able to figure out 8 x 7 (Clarke & Kamii, 1996).

Rational Numbers

A strong association exists between students' knowledge of fraction sizes and their competence in fraction calculations (Siegler et al, 2011). Students' ability to represent the size of fractions on number lines has been associated with success in solving problems involving fraction arithmetic (Siegler et al, 2011). Students can be supported in doing this by progressively aligning whole numbers on various number lines and receiving feedback on their efforts, e.g. placing 2, 9 and 24 on a number line from 0 to 100, and then placing 20, 90 and 240 on a number line from 0 to 1,000 (Thompson & Opfer, 2010).

Irish primary mathematics textbooks typically emphasise a part-whole construct of fractions and use area model representations (Charalambous et al, 2010). Although, part-whole representations are accessible and concrete, their limitations arise with improper fractions, negative fractions, and fractions with large numerators or denominators (Siegler et al, 2011). Furthermore, varying fraction constructs and representations (see Figure 3) helps focus on the numerical magnitudes of fractions, making it easier to connect learning fraction magnitudes with learning whole numbers, where the focus is typically on quantity or size (Siegler et al, 2011).

The primary mathematics curriculum orders the introduction of rational numbers as fractions, decimals, percentages (Government of Ireland, 1999). Some other countries follow this order (e.g. United States and Israel) but in France teaching decimals precedes fractions (Resnick et al, 1989). Some researchers suggest reversing the order in which rational numbers are introduced (Moss & Case, 1999). Deciding the order in which students should learn various rational number formats should be influenced by ease of learning, speed of overcoming misconceptions, and transfer of knowledge (Tian & Siegler, 2018). However, the shortage of comparable research in fractions and decimals and the paucity of research on percentages mean that research evidence for changing the order is currently insufficient (Tian & Siegler, 2018). Nevertheless, working across all three formats may help students grasp key rational number concepts.

One emerging, tentative recommendation from research is that fractions be introduced to children before multiplication and that fraction numbers be used in developing the concept of multiplication (Simon et al, 2018).

Interpretations of Fractions

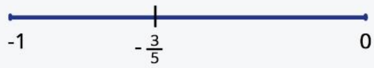
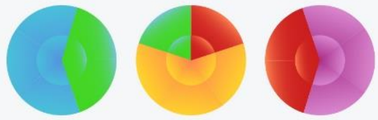
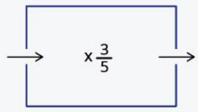
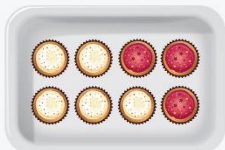
FRACTION INTERPRETATION	REFERENT WHOLE UNIT	FRACTIONAL PART SHOWN	ILLUSTRATION
PART-WHOLE (PROPER)	1 CIRCLE	$\frac{3}{5}$	
PART-WHOLE (IMPROPER)	1 CIRCLE	$\frac{5}{3}$	
LINE SEGMENT	1 LINE SEGMENT	$\frac{3}{5}$	
NUMBER (POSITIVE) Point on the number line	1	$\frac{3}{5}$	
NUMBER (NEGATIVE) Point on the number line	1	$-\frac{3}{5}$	
QUOTIENT (FRACTIONS AS DIVISION – DIVIDE 3 PIZZAS EQUALLY AMONG 5 CHILDREN)	1 PIZZA	$\frac{3}{5}$	
OPERATOR Scaling	1	$\frac{3}{5}$	
RATIO (PART TO PART – PINK TO WHITE)	8 CAKES	3:5	
RATIO (PART TO WHOLE – PINK TO ALL CAKES)	8 CAKES	3:8	

Figure 3. Interpretations of fractions.

Processes and Number

Reasoning

Encouraging students to reason requires teachers to listen responsively (Empson & Jacobs, 2008; Takker & Subramaniam, 2014) when presented with moments of contingency (Rowland et al, 2015). Frequently such moments arise when students answer questions, make spontaneous remarks, or offer unexpected incorrect responses (Rowland et al., 2015). Engaging in mathematical reasoning through generating conjectures, justifying ideas, or generalising ideas may elicit such student interactions (Takker & Subramaniam, 2019).

Difficulties arise for senior primary students because the mathematics is more complex and previous learning can interfere with new learning. In decimals, students may overgeneralise their understanding that thousandths are smaller than hundredths which are smaller than tenths to believe that numbers with one decimal place are automatically larger than numbers with two decimal places, which are in turn larger than numbers with three decimal places (e.g. believing that 0.6 > 0.64 > 0.642); other students may mistakenly apply knowledge from fractions that ½ > $^{1}/_{35}$ and erroneously conclude that 0.2 > 0.35 (Steinle & Stacey, 2004); others may apply whole-number knowledge and conclude that the length of the decimal string determines the size of a number (Resnick et al, 1989).

Although some misconceptions, at least temporarily, seem inevitable when learning mathematics, curriculum designers and teachers need to be mindful that the sequence in which concepts are introduced may play a role in promoting or reducing potential student misconceptions (Resnick et al, 1989).

Algorithms

Researchers have documented mixed findings about whether and when standard written algorithms for addition, subtraction, multiplication and division calculations should be taught. One study from France suggests that applying standard algorithms may have an adverse effect on children's calculations. However, the authors decline to advise against teaching standard algorithms noting that many factors must be considered in making such a decision. The closest they come to making a recommendation is to hint at delaying the introduction of standard written algorithms (Fischer et al, 2019).

Key Messages

Limitations

Number is a substantial and evolving strand in mathematics education research. In order to synthesise key findings for this paper, some omissions were required and some topics were prioritised over others. The paper does not cover the impact of technology – such as the calculator – on the teaching and learning of number.

Most mathematics education research available for compiling the report was conducted in settings outside Ireland and differences across countries due to factors like curriculum sequence or language of instruction (Resnick et al, 1989; Stigler and Hibert, 1999) cannot be ruled out. Therefore, if curriculum adaptations are proposed based on these recommendations, discussions or pilot studies with Irish teachers and students should be considered. Finally, treating number as a stand-alone topic risks duplicating or omitting material that overlaps with other curriculum strands.

Key Recommendations

- Develop place value understanding throughout all primary school years.
- Help students to understand the links among place value, division, fractions, ratio and proportion, area and volume.
- Reduce the focus on presenting multiplication as repeated addition and guide students towards a multiplicative perspective on number.
- Use fraction constructs and representations that convey magnitude
- Present the equals sign as indicating an equivalent relation and not a requirement to calculate.
- Encourage students to articulate and test conjectures about numbers and operations.

Tentative Recommendations

- Introduce percentages first followed by decimals and then fractions; introduce fractions before multiplication. Such research is emerging and further supporting evidence is desirable.
- Consider the use of multiple labels for decimal numbers such as 0.41 (zero point four one; 4 tenths and 1 hundredth; 41 hundredths).

Glossary

Cartesian product: The product of two sets A and B where the elements of A are matched with the elements of B. For example, 3 shirts (x,y,z) and 3 shorts (1, 2) could be matched as A X B ={ (x,1), (y,1), (z,1) (x,2), (y,2), (z,2)}

Conceptual field: A conceptual field consists of integrated situations and concepts in a mathematical topic. It facilitates the classification of mathematical relationships that help students move from using intuitive strategies to developing intuitive knowledge which is transformed into explicit knowledge (Vergnaud, 1988, 2009).

Multiplicand: A quantity that is multiplied by another number.

Multiplier: A quantity by which another is multiplied.

Product: A quantity produced by multiplying the multiplicand by the multiplier.

Relational thinking: Relational thinking occurs when students look at an equation in its entirety rather than as components to be calculated. In such circumstances students conceive of the equals sign as a relational symbol rather than a signal to calculate (Stephens & Armanto, 2010).

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