LEARNING AND TEACHING PRIMARY MATHEMATICS

An addendum to NCCA research reports 17 and 18

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Introduction

Important considerations for a high-quality mathematics curriculum for middle/upper primary pupils are explored in this paper. It builds on the two research reports published by the National Council for Curriculum and Assessment (NCCA) in 2014 to underpin a redeveloped mathematics curriculum for 3 - 8 years (Dooley, Dunphy, Shiel et al., 2014; Dunphy, Dooley, Shiel et al., 2014). In the first report, Report no.17 (referred to hereafter as RR17), the focus was on the theoretical aspects of mathematics education for young children; the second report, Report no.18 (RR18), was concerned with pedagogical implications. While these reports are concerned with mathematics education in the early years, their content is highly relevant to the older age group. Importantly both are premised on a view of mathematics espoused by Hersh (1997): "mathematics as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (p.xi). Moreover, all children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking.

An emerging curriculum model, incorporating mathematical proficiency, mathematization, goals, learning paths, narrative descriptors, and learning outcomes, is presented in RR18 and remains suitable for older primary learners. This model, shown in Figure 1 below, is inserted in a frame here to emphasise the interconnected and dynamic relationship among these elements.

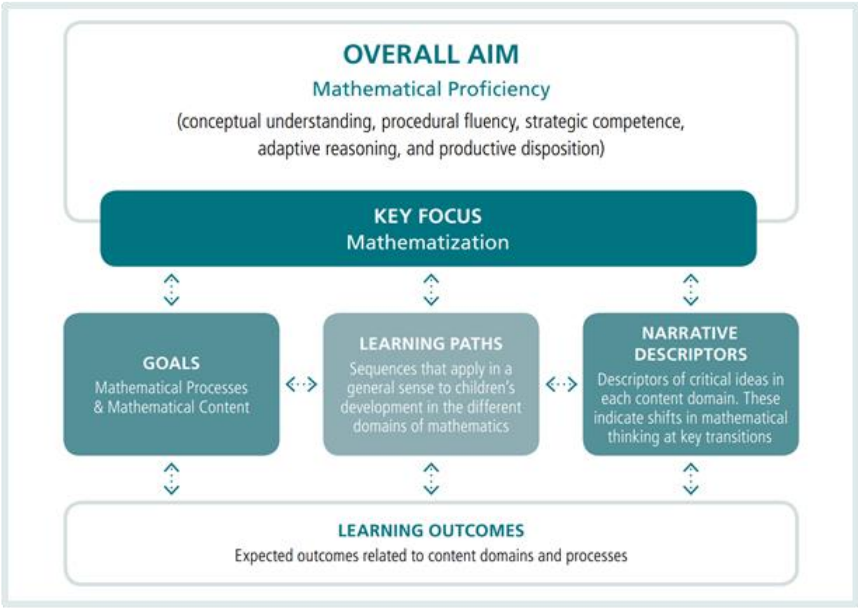

Figure 1: Curriculum Model. Adapted from *Mathematics in early childhood and primary education (children aged 3-8 years). Research Report No.18: Teaching and learning*, (p.11), by T. Dooley, E. Dunphy, G. Shiel, et al., (2014). Retrieved from https://www.ncca.ie/media/2147/ncca_research_report_18.pdf.

Consistent with many international mathematics policy documents, the overall aim of the curriculum is the development of mathematical proficiency which comprises five interdependent and interconnected strands – conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (National Research Council (NRC), 2001). As outlined in RR17, learners' mathematical proficiency develops as their mathematical experiences expand and deepen throughout their early, primary and post-primary years and beyond. Mathematization where children have the opportunity to reinvent mathematics (Freudenthal, 1973) is seen as critical in developing mathematical proficiency. Curricular goals relate both to processes and content. The five content domains are the same as those found in the 1999 Primary School Mathematics Curriculum (PSMC) - Number, Algebra, Shape and Space, Measures, and Data (Gov. of Ireland, 1999). The processes - communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting - can also be found in the 1999 PSMC. However, as described in RR17, content objectives have received more attention than processes in the implementation of the 1999 PSMC. In the redeveloped curriculum it is expected that, in line with a sociocultural perspective, processes will be foregrounded while content areas will also be specified. Learning paths are the sequences that apply in a general way to children's development in the different mathematical domains. Critically these learning paths are seen as provisional and non-linear, not related to class-level (e.g., 3rd), and are dependent on learners' mathematical experiences. In a similar vein, while learning outcomes at a general level should be specified in curriculum documentation, the co-construction of learning outcomes by teachers and children is emphasised.

In RR18 the features of good pedagogy are identified – consistent with the theoretical perspectives espoused (cognitivist/socio-cultural/constructionism) these pedagogical features are grouped under three headings: People and Relationships, the Learning Environment and the Learner. All of the features listed in this report are relevant for older children, although, in keeping with the relative maturity of the child, play might be interpreted somewhat differently. In a similar way, the five overarching pedagogical meta-practices - promotion of math talk, development of a productive disposition, emphasis on mathematical modelling, use of cognitively challenging tasks, and formative assessment - remain key to the promotion of mathematical thinking and understanding, and the aim of developing mathematical proficiency in older children. Given the centrality of pedagogy as highlighted in RR18, it is hardly surprising that the recommendations concerning teacher preparation and development made in the research reports are also pertinent.

Notwithstanding the relevance of most of RR17 and RR18 for middle/upper primary pupils, there are some matters that need further consideration or emphasis in the redevelopment of the primary mathematics curriculum for this group of learners. For example, since the publication of the research reports, new data concerning primary mathematics learning and teaching are now available. These data are considered under three headings: curriculum, performance, policy and social. As children progress through primary school, many will think in a more symbolic manner about objects that they experienced either in real-life or in imagined worlds on previous occasions. Tall (2008) explains it thus:

> School mathematics builds from embodiment of physical conceptions and actions: playing with shapes; putting them in collections; pointing and counting; sharing; measuring. Once these operations are practiced and become routine, they can be symbolised as numbers and used dually as operations or as mental entities on which the operations can be performed. (p.9)

This shift from embodied to symbolic thinking deserves particular attention and, especially, how children's informal learning experience might become the basis of conceptual understanding. Problem-solving situations that are of interest to learners are explored for their potential to develop children's mathematical thinking and to support the symbolisation process. Moreover, the meta-practices identified in RR18 are exemplified for the older age group. Other issues addressed in this paper relate to STEM, inclusion, and frameworks for the analysis of teaching. It should be acknowledged that, while the focus of the paper is on older primary school children, the general ideas explored are relevant for all learners, including the younger age group.

Context

Performance

Since the research reports were published, results of the 2014 national assessments of English Reading and Mathematics (NAMER) have become available (Shiel, Kavanagh & Millar, 2014; Kavanagh, Shiel, Gilleece, & Kiniry, 2015). It was found that, compared with the assessments conducted in 2009, there were significant improvements in the performance of second and sixth-class pupils on content and processes (the exception being that of the performance of second-class pupils on data items where the difference was not significant). In second class, pupils performed best on items assessing the skill "understand and recall", while pupils in sixth class achieved the highest scores on items assessing "reasoning". There was a reduction in the proportion of lower achieving children, and the performance of higher achieving pupils had improved. In both classes, the items involving "apply and problem solve" were found to be most difficult. Problem solving was identified by teachers as an area in which Continuing Professional Development (CPD) was required and was also regarded by parents of sixth-class pupils as an area of concern. In terms of materials used, it was found that 91.5% of sixth-class pupils were in classrooms where the textbook was used most or all days; digital resources were used in about 35% of sixth-class mathematics lessons on most or all days although the authors state that it is not clear where these resources were used by the teacher, pupils or both. Interestingly, regular access to calculators and manipulatives in sixth class was associated with a lower mean score, indicating that these were used by pupils who were considered to be low attaining in mathematics. The findings of Trends in International Mathematics and Science Study (TIMSS) administered to children in fourth class in over 50 countries in 2015 yield a similar picture (Clerkin, Perkins, & Cunningham, 2016). Pupils in Ireland did better on number and less well on geometric shapes and measures, when compared to their overall performance. They displayed a relative strength on “knowing” (e.g., recall of facts) and a relative weakness on “reasoning”, when average performance in these domains was compared to their overall performance. This is interesting given pupils’ performance on reasoning items in NAMER 2014. However, reasoning skills as interpreted by TIMSS are equated with the application of more complex procedures in unfamiliar scenarios to solve a problem (and therefore similar to “apply and problem-solve” in the 1999 PSMC). In mathematics lessons, pupils were most frequently asked to listen to their teacher explaining new content (73% of pupils in “every or almost every” lesson), and to listen to the teacher explaining how to solve problems (57%) (Clerkin, Perkins, & Chubb, 2017). Another finding that emerged was that while lower attaining pupils performed relatively well, those in the high attaining category did not do as well as their counterparts in other countries where the mean performance was similar to Ireland. There is much in the findings of both NAMER 2014 and TIMSS that is encouraging – notwithstanding this, matters related to relative difficulty pupils experience in items assessing higher-order thinking, and the performance of high-attaining children remain causes for concern.

Policy

In 2015 the digital strategy for schools was published. This set out strategic actions to support the integration of digital learning objectives within education policy and curriculum initiatives. In this area, NCCA’s work on the Coding in Primary Schools initiative is nearing conclusion. Upon completion, and alongside the continued work of the wider-redevelopment of the primary curriculum, an outline of the place of digital technology in the curriculum is expected to be provided. In conjunction with this, the PDST is currently building the capacity of teachers to teach basic coding and programming to children.

Recently a STEM Education Review Group (STEMerg) conducted a comprehensive review of STEM Education in Irish schools (STEMerg, 2016). Its vision is “to provide students in Ireland with a STEM education experience of the highest international quality; this provision should underpin high levels of student engagement, enjoyment, and excellent performance in STEM disciplines” (p.6). Within the report, issues such as the preparation of teachers to teach STEM subjects, CPD, inquiry-based learning approaches, engagement by students in STEM subjects and the promotion of STEM careers are addressed. Importantly STEM is conceptualised as follows:

> The use of the acronym has come to signify more than a simple list of related disciplines. Effective STEM education helps the learner to develop the disciplinary knowledge (e.g., Biology, …), the skills (e.g., problem-solving, design, IT skills), and habits of mind (e.g., inquiry, evidence-based reasoning, logical thinking) associated with STEM disciplines. In this report, Mathematics is viewed as a fundamental discipline since it underpins all of the other STEM disciplines. (p.13)

Of particular relevance to this paper is the importance attached to mathematics in the STEMerg report and the recognition that STEM experiences of primary pupils are significant for choices that they make regarding the adoption of STEM subjects in post-primary and third-level contexts.

Curriculum

Work has commenced on the redevelopment of the primary school curriculum. This work comprises research, working with schools, and consulting with wider stakeholders in the form of seminars and online portals. The new Primary Language Curriculum and the draft specification of the Primary Mathematics Curriculum for 3-8 year-olds both centre around learning outcomes. The outcomes are supported by suggested learning experiences as portrayed in progression continua. These learning experiences combine processes (entitled "elements") and content. It is anticipated that developments in the new Primary Mathematics Curriculum for the middle/senior classes will build on this work.

Since 2014, the Junior Cycle features revised subjects and short courses combined with new approaches to assessment and learning. The revised mathematics syllabus has been in place in schools since September 2018. In line with the primary school curriculum, the aim of the revised Junior Cycle mathematics syllabus is the development of mathematical proficiency. Moreover, there is an emphasis on students exploring interconnected mathematical ideas in collaborative settings, and on reasoning processes through open-ended investigations. There is also a greater focus on formative assessment than heretofore. A review of Senior Cycle is currently being undertaken by NCCA and features work with many stakeholder groups to generate a shared vision for a redeveloped senior cycle.

Social

In RR17, there is allusion to the significant demographic changes that have occurred in Ireland since the introduction of the PSMC in 1999. The particular difficulties in learning mathematics found by children who speak a language other than English or Irish at home were described. Also addressed were the challenges faced by pupils attending schools designated as disadvantaged (and thus part of the Delivering Equality of Opportunity in Schools (DEIS) programme). The findings of NAMER 2014 indicate some progress in both areas. For example, there was an improvement in the performance of children attending DEIS schools but this was consistent with schools in general – with the exception of second-class pupils in DEIS band 2 schools who did relatively well, their performance was still below national standards (Shiel et al., 2014). While second class pupils who mostly spoke a language other than English or Irish at home had a significantly lower mean mathematics score than those who spoke mostly English, this was not the case in sixth class where those who spoke mostly English at home had no advantage in mathematics over those who did not (Kavanagh et al., 2014). However, girls in both second and sixth classes in NAMER 14 had significantly lower mean scores than boys on the Measures content area (which includes several problems) and on the "apply and problem solve" process skill – this contrasted with the findings of NAMER 2009 where there were no significant gender differences in the content or process subscales. In 2014, girls also scored less well than boys on a measure of mathematical self-concept. In TIMSS 2015, while fourth-class boys in Ireland significantly outperformed girls on Geometric Shapes and Measures, there were no significant gender differences across the three cognitive domains, "knowing", "applying" and "reasoning". Alongside this the Growing up in Ireland (GUI) study has produced some pertinent findings on the transition from primary to second-level education (Smyth, 2017). In particular, maths test scores and attitudes to maths at the age of nine were predictive of engagement with the subject in later years, a finding that is consistent with the thrust of the STEMerg report mentioned earlier. In RR17, the importance of a high-quality early education is highlighted as a "critical factor in ensuring that the mathematics potential of all children is realised and that existing equity gaps are closed" (p.21). It is vital that a high-quality mathematics education continue throughout the primary years for all pupils.

Powerful Mathematical Ideas [for primary children aged 8-12 years]

In RR17, consideration is given to the "Big Ideas" of mathematics. It was described that teachers' understanding of big ideas in children's mathematical learning are seen, especially in the US, as critical to the development of young children's mathematical understanding. It was also outlined that while there seems to be general agreement on big ideas as ideas that connect various concepts and procedures within and across domains (Baroody, Purpura & Reid (2012) cited in RR17, p. 72), there is less agreement on what these big ideas might be. From a cognitivist perspective, the focus of big ideas is on content whereas from a sociocultural perspective, there is a greater emphasis on processes (see Table 4.1 on p. 73, RR17). Less attention has been given to big ideas for older primary children than is the case for their junior counterparts. This can possibly be explained by the fact that the extremely influential work of Clements and Sarama (2009) on learning trajectories (based on goals/big ideas) is largely although not exclusively confined to pre-primary and earlier primary years.

In the second and third editions of *Handbook of international research in mathematics education* (English et al., 2008; English & Kirshner, 2015), "powerful" (rather than "big") ideas for elementary children are explored. In a similar vein to Perry and Dockett (2008) (see RR17), Langrall, Mooney, Nisbet and Jones (2008) see these ideas as those that transcend content domains. Noting that the focus of much of the research for this age group is on models that describe children's thinking processes within the various content domains, (Number, Geometry, Algebra, Data, Measures), they searched within the literature for ideas that recurred within and across these domains. The five themes that they identified are next described and elaborated.

Unit, Iteration, Composition and Decomposition

This idea applies particularly to the case of Number (e.g., base-ten place-value system, addition, number operations and fractions), Measures and Geometry. For example, Tzur (2004) suggests that fundamental to an understanding of fractions is the idea of a unit and iteration, that is, if a given whole is partitioned equally (say *m* equal pieces), a unit of a particular size relative to the size of the given whole is produced (1/m). The iteration of this unit *m* times will reproduce the original whole unit (*m/m*). Outhred and Mitchelmore (2000) argue that understanding of rectangular area is based on the learner's grasp of a row as an iterable unit as well as the relation between the size of the unit and that of the rectangle. Clements, Wilson, and Sarama (2002) assert that the ability to describe and visualise the effects of composing and decomposing geometric shapes underpins the ability to think geometrically.

Generalisation and Formalisation

The importance of generalisation is signalled in RR18 where it is described as "a shift in thinking from specific statements to more general assertions" (p.62). While formal generalisation is more often associated with older students (e.g., manipulation of algebraic formulae in post-primary school), the capacity to generalise is found in very young children. Carraher, Martinez, and Schliemann (2008) emphasise that generalisations stem from rich and experiential activities, that is, young children's generalisations arise from their considerations about how physical quantities change or remain the same as a result of actions and operations. Such thinking has relevance to all content domains (e.g., what is common to all parallelograms etc.). Indeed Mason (2008) argues that a lesson without the opportunity to generalise cannot be considered to be a mathematics lesson.

Variation and Expectation

Watson, Callingham, and Kelly (2007) describe the idea of central tendency as expectation and that of spread as variation. Admitting that these ideas at a technical level are probably more relevant for older (post-primary) students, they argue that an understanding of them is within the reach of the primary age group. Children are most likely to experience expectation with respect to data and chance as probabilities or the random distribution of outcomes; they link variation, on the other hand, with uncertainty, anticipated and unanticipated change and outliers. Watson et al. suggest that an ability to engage in proportional reasoning is key to sophisticated understanding of these ideas. They also highlight other content domains as contexts for exploring and understanding them, e.g., measurement and numbers. They emphasise the relevance of pattern, normally associated with algebra, to an appreciation of data and chance matters.

Equality

At the heart of many of the content domains of mathematics is the notion of equality or "quantitative sameness" (Langrall et al., 2008, p.116). Prediger (2010) distinguishes between two types of equality – operational equality and relational equality. Operational equality, in which the focus is almost entirely on asymmetric equalities such as 2x3x4= 24, is usually emphasised in primary school mathematics. However, relational equality - signified by a symmetric use of the equal sign - is a foundational concept linking arithmetic and algebra. Pupils who develop relational understanding of the equal sign realise that it indicates the sameness of expressions or quantities signified by each side of an equation, e.g., 4x6 = (4x3) + (4x3).

Representation

According to Langrall et al. (2008), the many forms of representation that children use are a means not only of communicating their thinking but also of organizing and advancing their thinking. This concept is discussed in RR18 where the examples of representation such as concrete manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories are drawings are outlined. Goldin and Kaput (1996) distinguish between internal representations which are the individual's mental models of a given situation and external representations such as graphs or words. They note the significance of the reciprocal relationship between the two forms:

> Of special importance are the two-way interactions between internal and external representations. Sometimes an individual externalizes in physical form through acts stemming from internal structures—that is, acts of writing, speaking, manipulating the elements of some external concrete system, and so on. Sometimes the person internalizes by means of interactions with the external physical structures of a notational system, by reading, interpreting words and sentences, interpreting equations and graphs, and so on. (p.401)

Heinze, Star,and Verschaffel (2009) claim that representational fluency, that is, the facility to use and flexibly switch between a wide range of representations, is a critical aspect of mathematical problem solving. Moreover, they suggest that a learning environment where students are presented with many representations of a particular mathematical concept or a principle or a situation and where they are encouraged to switch flexibly between them is considered to be conducive to supporting students' understanding and indeed fostering in them a productive disposition to mathematics. Further attention is given to some of these ideas in the section on modeling below.

Emerging Powerful Ideas

Langrall et al. (2008) recognise that the powerful ideas that they identified in their review are "only a beginning" (p.117) and that others will emerge with continuing research in different areas of mathematics education. In this regard it is interesting that recently, the powerful ideas described by Carraher and Schliemann (2015) centre around functions and relations. They contend that "fostering students' early understanding of functions and relations can deepen and integrate their knowledge about topics in the current curriculum, increase their ability to make mathematical generalizations, and better prepare them for a late, more formal introduction to algebra and functions." (p. 191). They cite examples from the domains of Number, Algebra and Geometry in support of their claim. For instance, they say that, from a relational point of view, every addition problem can be viewed as a subtraction problem; equations can be regarded as statements expressing the conditions under which two functions give the same output for a given input. It can be seen that Carraher and Schliemann' s powerful ideas encapsulate some of those outlined by Langrall et al., in particular the focus on generalisation, representations and equality.

While it can be expected that other powerful mathematical ideas will emerge in time, their importance lies not only in planning and implementing learning and teaching activities but also in directing attention to the capacity of primary children to deal with big and important mathematical ideas. In other words, given the opportunity to engage in problem solving and to build on their informal mathematical knowledge, primary children are capable of understanding rich mathematical ideas that were once preserved for older students (Langrall et al., 2008). It is not that content that has hitherto been considered too difficult for primary children should be emphasised in the redeveloped curriculum, but that consideration should be given to the possibilities for deep learning of mathematical content that arise when new pedagogies for mathematics are implemented.

Good Mathematics Pedagogy

Theoretical perspectives on mathematics teaching and learning give rise to models of teaching (e.g., Simon, 1995; Renert and Davis, 2010). One of the key messages in RR17 is that "sociocultural perspectives, cognitivist perspectives, and a constructionism perspective each offer insights which can enrich … understanding of issues related to the revision of the curriculum … by providing key pointers to each of the elements of learning, teaching, curriculum, and assessment" (p.56). This key message is also relevant for middle/upper primary pupils. In RR18, three principles of learning with respect to mathematics – arising from the theoretical perspectives above – are presented, that is,

- Teachers must engage children's preconceptions.
- Understanding of mathematics requires factual knowledge and conceptual frameworks.
- A metacognitive approach enables children to monitor their own learning and environment.

Features of good pedagogy that derive from these principles were grouped under the headings *People and Relationships*, *The Learning Environment* and *The Learner*. For example, a feature related to the first of these is that the diverse cultures of children and their families are taken seriously and treated as classroom resources; one of the features classified under the Learning Environment is that learners are given opportunities to engage in justification, argumentation and generalisation; and a feature related to the Learner is that children's reasoning is at the centre of instructional decision making and planning. It is now known that children learn most effectively when they are actively involved in the process and when it takes place in context (NRC, 2000). It is therefore not surprising that, in RR18, play is given particular emphasis as a context for the promotion of mathematical thinking and learning in young learners. Play is described as that in which children spontaneously engage (and which can serve as the basis of teacher-guided activity) and also as play with mathematics itself. Moreover, considerable emphasis is placed on playful approaches to engaging young children in mathematically related activities and discussions While these ideas should not be overlooked as an important vehicle for learning in older children (Isenberg & Quisenberry, 2002), the main focus in the mathematics literature for this age group is on engagement in situations which have potential to develop children's mathematical thinking (Fosnot & Dolk, 2001) – more generally referred to as problem-solving situations. As will be described later, these can offer rich ground for "mathematical play" (Holton, Ahmed, Williams, & Hill, 2001).

Problem-solving Situations

Problem solving has long been regarded as an important goal of mathematics education. D'Ambrosio (2003) outlines how in the mid to late 1800s, educators viewed mathematical problem solving as application of the principles of the subject. They therefore believed that students should be shown mathematical procedures, and then be given an opportunity to practice the procedures and apply them to word problems. He remarks that vestiges of this thinking remain today. However, there are now much broader conceptions of problem-solving and the role it plays in mathematics teaching and learning. Stanic and Kilpatrick (1988), in their overview of problem solving, point to three themes that have characterised problem-solving in the school mathematics curriculum:

(i) problem solving as context, based on the idea that problems and problem-solving are a means of achieving other ends such as motivation, concepts and skills etc.;

(ii) problem solving as skill where it is viewed as one of a number of skills to be developed through engagement with mathematics; and

(iii) problem solving as art which is a perspective, proposed in particular by Polya (1981) and Dewey (1938), holding that the solving of problems that have been identified as important or that are posed by learners themselves, is at the core of mathematics.

With its emphasis on mathematics as a creative pursuit, it is hardly surprising that (iii) finds much favour by key researchers in mathematics education. In the 1999 PSMC problem solving is viewed as one of the five skills developed through engagement with content. It is also one of the principles of teaching and learning mathematics:

> Developing the ability to solve problems is an important factor in the study of mathematics. Problem-solving also provides a context in which concepts and skills can be learned and in which discussion and co-operative working may be practised. Moreover, problem-solving is a major means of developing higher-order thinking skills. These include the ability to analyse mathematical situations; to plan, monitor and evaluate solutions; to apply strategies; and to demonstrate creativity and self-reliance in using mathematics. Success helps the child to develop confidence in his/her mathematical ability and encourages curiosity and perseverance. Solving problems based on the environment of the child can highlight the uses of mathematics in a constructive and enjoyable way. (Gov. of Ireland, 1999, p.14)

Here there is particular focus on "problem-solving as context" and "problem solving as skill". Recently the emphasis in the use of problem-solving in international mathematics curricula is that of teaching mathematics through problem solving. According to D'Ambrosio (2010), this aligns closely with (iii) above although it also contains elements of (i). It is one of the themes mentioned in RR18 – "through engaging in problem solving, children not only learn problem-solving strategies but also deepen their understanding of mathematics" (p.70) and should be reflected in the learning experiences of middle/upper primary pupils. This, however, requires a reconceptualisation by teachers and the wider community of the role of problem-solving in the mathematics.

English and Gainsburg (2015) describe how traditionally problems were regarded as problems if their solution was not obvious, but are now regarded more broadly, for example, as activities that challenge and extend a person's thinking. Various problem classification systems have been used to analyse mathematics textbooks, e.g., routine versus non-routine, open-ended vs close-ended, traditional vs non-traditional (the latter incorporating projects, journals, puzzles, and problem-posing problems); application vs non-application among others (Zhu & Fan, 2006). While open-ended, non-routine problems are generally regarded as being more in keeping with children's sense making and the development of their mathematical thinking and understanding, research findings on this are not consistent. For example, arising from the research of Carraher, Carraher and Schliemann (1985) into the street mathematics of Brazilian children, there has been an interest in the use of real-life or authentic mathematical problems in classrooms. However, the success of this has been limited. One reason is that it is difficult to reproduce authentic situations in classrooms (Ainley, Pratt & Hansen, 2006). Another is that children from lower-income families perform less well in a contextualised mathematics curriculum, mainly because they fail to ignore the context in a way that children from high-income families tend to do (Cooper & Dunne, 1998). Moreover, the authenticity often derives from an educator's or textbook author's reality rather than that of pupils.

Arguing from a social justice perspective, Gutstein (2003) talks about the need for mathematics to be culturally relevant, and advocates for real-world mathematics projects that are connected to the lives and experiences of the learners themselves. This calls for a relational, sensitive pedagogy in which teachers are aware of and use learners' knowledge, beliefs, values and experiences as resources for teaching mathematics (e.g., Bartell, 2012).

While the value of problem-solving is recognised in curricula and by schools and teachers, its enactment in classrooms proves challenging internationally (English & Gainsburg, 2015). One explanation is that mathematics is often viewed as a bank of knowledge to be transmitted from one generation to the next (Ernest, 1989). The integration of problem-solving into the mathematics classroom requires, according to Santos-Trigo (2007), a conceptualisation of mathematics as a set of problems or dilemmas that are solved through the use of mathematical resources:

> In order to recognize and value a learning approach based on problem solving, we must identify key or relevant principles that need to be clear when following this approach to learn and solve mathematical problems. I argue that an overarching principle that characterizes any problem solving approach to construct or learn mathematics is that researchers, teachers and students conceptualize the discipline as a set of problems or dilemmas that need to be examined and solved through the use of mathematical resources. Thus, problem solving is an inquiry domain in which learners are encouraged to pose and pursue relevant questions. To inquire means to formulate and pursue questions, to identify and investigate dilemmas, to search for evidence or information, and to present and communicate results. It means willingness to wonder, to explore questions and to develop mathematical understanding within a community that values both collaboration and constant reflection. A mode of inquiry involves necessarily the challenges of the status quo and a continuous reconceptualization of what is learned and how knowledge is constructed. (Santos-Trigo, 2007, p.526)

Here we can see "problem-solving as art" espoused. There is reference to the classroom as a community where pupils listen to and respond to each other's ideas. It is similar in vein to Hiebert and Wearne's (2003) recommendation that all mathematics, even routine aspects, be 'problematic' for learners. For the teacher, this means posing problems (or using problems posed by the children themselves) that are just within the reach of learners, allowing them to struggle to find solutions and then reflect on their own and others' solution methods. Importantly, creative problem-solving is linked with the idea of mathematical play, described by Holton et al. (2001) as "that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas" (p.403). They say that these ideas might not necessarily help to solve the problem at hand but that learning what is not applicable is as important for learners as finding a correct solution. As such, mathematical play is important for learners of all ages.

The features of good pedagogy listed in RR18 are relevant for a learning environment where problems of interest to the pupils themselves, mathematical play and pupils' own problem-solving strategies are leveraged and valued. For example, the feature 'classroom activity and discourse focus explicitly on mathematical ideas and problems' allows for the time and space to children to engage in mathematical play. Three of the features specifically mention play, that is,

- A wide range of children's everyday activities, play and interests are used to engage, challenge, and extend their mathematical knowledge and skills.
- Children experience opportunities to learn in teacher-initiated group contexts, and also from freely chosen but potentially instructive play activities.
- The potential of everyday activities, such as cooking, playing with mathematical shapes and telling the time is recognised and harnessed.

The first of these should be retained as play might indeed be a context for children's problem-posing and problem-solving activities. It is recommended that the others be amended to

- Children experience opportunities to learn through creative and playful approaches where they generate ideas around the solution of problems that are of interest to them.
- The potential of everyday activities is recognised and harnessed.

The updated list of features can be found in the appendix. There are implications for meta-practices which are next considered.

Meta-practices

Meta-practices are practices that shape and are shaped by other practices. For example, children's educational and social practices shape and are shaped by novel teaching approaches which, in turn shape and are shaped, by teacher education programmes (Kemmis, 2012). As such, meta-practices are complex, interconnected and fluid. In RR17, meta-practices that are particularly significant for the development of young children's mathematical understanding are identified. These include *promotion of math talk, development of productive disposition, emphasis on mathematical modeling, use of cognitively challenging tasks,* and *formative assessment*. The meta-practices identified in RR18 are also relevant for the older age group, although as with other examples cited above, some slight changes in emphasis are appropriate. Each of the practices will be briefly outlined below.

Promotion of Math Talk

In RR18, the various facets of math talk, including the importance of sustained interactions, verbal reasoning and argumentation, and the establishment of math-talk learning communities, are described. While the significance of children expressing and discussing their mathematical ideas was highlighted in the 1999 PSMC, there is evidence, as mentioned above, that senior class pupils participate on an infrequent basis in productive dialogue in mathematics lessons and that much of class time is taken up with teacher-led instruction (e.g., Eivers, Delaney, & Close, 2014). It is noted in RR18 that teachers face challenges in implementing math-talk communities in their classrooms. Some of these challenges arise from external influences such as curriculum coverage and summative assessment; others relate to mathematical knowledge for teaching (MKT) (Ball & Bass, 2003) which is critical for the recognition of children's errors or misconceptions, and for the identification of opportunities for extending their mathematical thinking. Still others might arise from confusion over the meaning and role of math talk and, more generally, language in mathematics teaching and learning. An overview of language and communication is given in Chapter 3 of RR17. Some of the main ideas relevant to the development of math talk are outlined below but, due to the scope of this paper, in a brief fashion.

Language is interpreted differently within mathematics education literature, for example, as verbal and non-verbal means of communication; as used by mathematics learners in different language communities; and the phraseology/vocabulary associated with the discipline (Morgan, Craig, Schuette & Wagner, 2014). Gutiérrez, Sengupta-Irving, and Dieckmann (2010) distinguish between the language *in* mathematics learning activity such as in the practices of argumentation, justification, generalisation etc. that constitute mathematical thinking and the language *of* the mathematics classroom which is the way the language is regulated by teachers and students as in, for example, classroom interaction patterns. Gutiérrez et al. (2010) note that increased attention to the social theories of learning has led to an obscuration of the lines between language *of* and language *in*, due, in particular, to the ideas of social norms and sociomathematical norms in inquiry-based mathematics lessons. Social norms are the norms of classroom discussion that are not particular to mathematics, e.g., the expectation that a student should explain his/her thinking. Sociomathematical norms, on the other hand, are "the normative aspects of mathematics discussions specific to students' mathematical activity" (Yackel & Cobb, 1996, p.461). Examples include what counts as an acceptable mathematical explanation in the classroom community or what might be considered as mathematically elegant. The teacher plays a central role in the establishment of these norms which can vary considerably from one classroom to another. In particular, a teacher's MKT influences the sociomathematical norms developed in his/her classroom.

One's view of mathematical objects also has bearing on the prominence given to language as a basis for learning. It is often considered by researchers, educators and others that mathematics is made up of objects with an independent existence that can often be experienced through language (Morgan et al., 2014). Sfard, whose research has gained considerable traction in recent years, offers an alternative perspective. Arguing from a socio-cultural perspective, she views learning as participation in activities associated with the domain in question. In a Vygotskian vein, she describes thinking as a "dialogical endeavour" (Sfard, 2001, p.26) where one engages with oneself in discussion, argument and awaiting a response. She maintains that learning of mathematics is initiation into the discursive practices of mathematics. Therefore, combining the ideas of cognition and communication, she proposes a commognitive approach which "is grounded in the assumption that thinking is a form of communication and that learning mathematics is tantamount to modifying and extending one's discourse" (Sfard, 2007, p.565). For Sfard, discourse goes beyond language to include anything that constitutes communicating and has a bearing on its effectiveness. She argues that a teacher's role is to help modify learners' everyday discourse into a more mathematical discourse (Sfard, 2001). In particular, she claims that the mathematical discourse must develop in tandem with everyday discourse. This perspective is at variance with the "learning for understanding" approach in which learners encounter and use a mathematical idea before formalizing it (Gutiérrez et al., 2010).

There is a perception by some mathematics educators that language is a source of difficulty in mathematics learning. According to Morgan et al. (2014), this arises from the perception of mathematics and language as separate entities. They suggest that a different perspective on mathematics, that is "doing mathematics essentially entails speaking mathematically" (p.846) (a view that is prevalent in the research field) provides an alternative way of thinking about communication in the mathematics classroom. From this perspective, teachers and other relevant individuals need to think about communication in mathematics as doing mathematics and vice versa. It is a viewpoint that should permeate the redeveloped mathematics curriculum if language, including math talk, is to be foregrounded.

Development of a Productive Disposition

Productive disposition, defined as the "habitual inclination to see mathematics as sensible, useful, worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p.116) is one of the five interwoven strands that comprise mathematical proficiency, proposed in RR17 as a key aim of the revised curriculum. As described in the meta-practices section in RR18, it can be fostered in young children through participation and engagement in mathematically rich and worthwhile activities and through support from significant adults and peers. These approaches are also relevant for the older age group.

Classroom practices shape student disposition (Gresalfi, 2009). Therefore, in classrooms where the focus is on reward for finding correct answers, pupils will engage in a limited way with the mathematical content whereas if participation and effort are emphasised, they are more likely to persist with challenges. Moreover, dispositions are not fixed and can shift over time. The kind of practices that have a positive impact on disposition are difficult to pinpoint. As Gresalfi (2009) points out, the act by the teacher of praising a pupil for participation rather than correct solutions in a mathematics lesson is unlikely to affect the pupil's disposition if the system (e.g., summative assessment) rewards correct solutions. She acknowledges the need for different elements of the classroom to interact:

> Thus, the dispositions that students enact are not simply inherent capabilities, or learned responses, but rather are the result of both what happens in moments (e.g., the strength of an opportunity to participate) and how the accrual of those moments shapes the kinds of opportunities that are subsequently offered (e.g., emergent expectations of students). (p.366)

The idea of mathematical mindset cannot be overlooked in discussions on productive disposition. Contrary to popular notions that only some individuals "are good at mathematics", Boaler (2016) takes issue with the notion of a mathematical brain, claiming that "no-one is born knowing math, and no-one is born lacking the ability to learn math" (p.5). Building on Dweck's (2006) work, she provides evidence of the importance of a growth mindset - seen in people who believe that capacity to do things can be developed through dedication and hard work - in achieving mastery (of mathematics in this case). She argues that a fixed mindset where students believe that their intelligence is fixed is damaging both for those who attain high and low scores in mathematics. Some teaching ideas she gives for creating and maintaining a "growth mindset mathematics classroom" include encouraging and believing in all learners; valuing struggle and failure; praising effort rather than intelligence; teaching mathematics as an open subject where there is opportunity to explore and be creative, and where the emphasis is on depth rather than speed; teaching mathematics as a subject of patterns and connections; encouraging intuition and freedom of thought; and using manipulatives, technology and interesting problems. Such a classroom can also have positive effects on the mathematical performance of females who generally do better in a learning environment that promotes collaboration and depth of understanding than one that encourages competition (Boaler, Altendorff, & Kent, 2011). This is not surprising because, as will be outlined later in the paper, activities that are consistent with the development of a productive disposition resonate with those that promote an equitable curriculum.

Emphasis on Mathematical Modeling

Mathematical modeling[1], conceived as a bridge between informal understanding and the abstraction of more formal mathematical ideas, is the focus of much attention in the mathematics education literature (e.g., Kaiser, Blum, Ferri, & Stillman 2011). It can include the use of physical actions, spoken words, written words, symbols and images (e.g., graphs, diagrams and pictures) to represent

[1] Both spellings – mathematical modelling and mathematical modeling – are found in the literature. The latter spelling is used in RR18.

an everyday situation. Blum (2011) describes modeling as a bi-directional translation between reality ("the rest of the world") and mathematics. In RR18, the mathematical modeling as interpreted in the Realistic Mathematics Education (RME) and by English and her colleagues were examined. In the paragraphs below, the relevance of these perspectives for the later primary years is discussed.

In the RME approach, children engage with a problem that is realistic or at least real to them and, as they do so, they generate a model that can assist them with finding a solution (van den Heuval-Panhuizen, 2003). Fosnot and Dolk (2001) explain how children's initial models represent their interaction with the situation rather the mathematical situation itself. As pupils continue to engage in sense making of the problem or a series of related problems, their model(s) extends beyond representation of their interactions to more generalised models of strategies, that is they move from a model *of* a situation to a model *for* thinking about mathematical relations (Gravemeijer & Stephan, 2002). While the model is first closely bound to the context in which it arises, the general goal of modeling is the generation of models that can be used in different contexts. Thus the move from model *of* to model *for* is "a major landmark in mathematical development" (Fosnot & Dolk, 2001, p.81). Van den Heuval-Panhuizen (2003) describes this in relation the engagement by children in Grade 5 and 6 in problems involving percentage. The children first used a drawing to represent a context related to percentage; they then developed a bar model for percentage and later the bar model was used in a more generalised way as a tool to support thinking about percentage as an operator.

English and Sriraman (2010) describe modeling problems as "realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artefacts or conceptual tools that are needed for some purpose, or to accomplish some goal" (p. 273). At the first stage of their modeling cycle, learners are given a "model-eliciting" problem where they are required to develop a model to describe, explain or predict a given phenomenon or situation; they then engage in related problems ("model exploration" and "model-application") where they extend, explore, and refine the constructs developed in the first stage, that is models are re-used and generalised. An important aspect of the cycle is children's critical reflection of the model(s) they devise. One example of this can be found in English (2006) in which there is an account of the mathematical modeling of 6th grade children as they created, over the course of several lessons, a consumer guide for deciding the best crisps.

It can be seen that the focus of mathematical modeling, as described above, is the development by children of their own models of a particular situation that are later generalised for use in a range of related mathematical problems. This perspective differs from the focus on pre-constructed or commercial models that can often be found in classrooms, and in particular manipulative materials. While such materials can be helpful for students of all ages to construct knowledge, they do not in themselves imply a transfer of learning: often children are attracted to the nature of the materials without doing or seeing the mathematics within them (Askew, 2016). According to Moyer (2001, p.177):

> It is the mediation by students and teachers in shared and meaningful practices that determines the utility of the manipulatives. Therefore, the physicality of concrete manipulatives does not carry the meaning of the mathematical ideas behind them. Students must reflect on their actions with the manipulatives to build meaning.

She emphasises the student's internal representation of ideas, tested against external representations as core to the learning of mathematics. Drawing on a range of literature in the area, Sarama and Clements (2016) conclude that learners need to have access to multiple representations of situations (e.g., manipulatives, drawings, verbalisations, symbols etc.) – related, as stated earlier, to representational fluency – but that deeper and multifaceted experience with one or a few manipulatives is more conducive to development of mathematical thinking than is use of a wide variety of materials. They also argue that, like physical manipulatives, virtual manipulatives can enhance mathematical understanding if used in a comprehensive and well-planned way.

Use of Cognitively Challenging Tasks

Earlier in this paper the role of "mathematical play" in developing mathematical thinking was discussed. Integral to this kind of play is the notion of children generating ideas that are new to them. According to English and Gainsburg (2015) this calls for cognitively challenging tasks that have multiple points of entry, require high-level thinking and reasoning, and prompt a variety of solution approaches. In RR18, reference is made to the seminal work of Stein, Grover, & Henningsen (1996) on cognitively demanding tasks. English and Gainsburg (2015) contend that,

> Stein et al.'s criteria provide a pertinent basis for designing mathematics-curriculum problems that target 21st-century demands related to communication and other general problem-solving skills. For example, problems with high cognitive demand require students to explain, describe, and justify; make decisions, choices, and plans; formulate questions; apply existing knowledge and create new ideas; and represent their understanding in multiple formats. (p.326)

While there has been a particular emphasis in this report on the use of problems that are real for learners, it is important not to overlook seemingly routine tasks (e.g., 34 x 7) in the redeveloped mathematics curriculum. Hiebert et al. (1996) maintain that tasks are rendered problematic if they are treated by teachers and pupils as such. In developing this idea, Askew (2016) differentiates between "task", what teachers require children to do, and "activity", what learners do when carrying out the task. If the task is to be problematic, there is, he suggests, a necessity for a gap between task and activity, that is it should not be reduced to a set of steps or procedures. For example, in the case of 34 x 7, above, children should have the opportunity to generate, share, and reflect upon their own and their peers' solutions methods, rather than follow a set of pre-ordained steps. Henningsen and Stein (1997) maintain that the factors influencing the choice of task include a teacher's goals, his/her knowledge of subject matter, and his/her knowledge of students; factors impinging on activity are classroom norms, task conditions, a teacher's instructional disposition and students' learning dispositions. In their study of the implementation of tasks in four second-level classrooms (students aged 12-15 years), they found that tasks are likely to have a high level of cognitive demand if they build on students' prior knowledge and if there is an appropriate amount of time – neither too little nor too much – for the students to engage at a high level. Supportive teaching behaviours included scaffolding and consistent pressing of students for meaningful explanations. Henningsen and Stein (1997) also looked at the cause for various types of decline of tasks over the lesson period. For example, amongst the factors that caused decline of the task into using procedures without connection to concepts, meaning, and understanding were removal of challenging aspects of the task (e.g., through teacher provision of explicit procedures), a focus on the correct solution, and inappropriate amount of time. Decline of tasks into activity with no mathematical substance was often due to unsuitability of the task or classroom management issues. It can thus be deduced that cognitive challenge is not necessarily inherent in a task but rather is dependent on the way it is set up and implemented by the teacher over the course of a lesson or lessons.

Formative Assessment

Assessment is an integral part of effective teaching and learning and, as elaborated by Lysaght, Scully, Murchan, O'Leary and Shiel (2019), it can manifest itself in various types or forms. An in-depth treatment of assessment of mathematics in the primary school is outside the scope of this paper and, for a more thorough account, the reader is directed to a chapter dedicated to the topic in RR17 and also to the above-mentioned paper by Lysaght et al. (2019). Formative assessment is given some attention here as it plays a crucial role in classrooms that value learners' thinking and problem-solving processes and, as such, is identified in RR18 as an important meta-practice. Black and Wiliam (2009) define formative assessment as follows:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited. (p.9)

They identify five forms of formative assessment: sharing success criteria with learners; classroom questioning; comment-only marking; peer- and self-assessment; and formative use of summative tests. Other means through which evidence of a student's progress can be gathered included in RR17 and RR18 are observations, tasks, interviews, conversations and pedagogical documentation, all of which are relevant with the older age group. However, Heritage et al. (2009) claim that, on the basis of a study they conducted with 6th grade teachers in the US, teachers are more competent at making inferences about learners' levels of understanding than at deciding on appropriate courses of action to take. This seems to be the case in Ireland also (Lysaght, 2015). Critically, a teacher's MKT is a critical factor in effective formative assessment.

Lysaght et al. (2019) speak of the importance of a "formative assessment classroom culture in which "insights" (rather than test scores) feed into and influence the nature and direction of learning and teaching in real time" (p.11). Of particular relevance to such a culture are the opportunities for formative assessment that arise from "the creation of, and capitalization upon, 'moments of contingency' in instruction for the purpose of the regulation of learning processes" (Black & Wiliam, 2009, p.10). Moments of contingency occur when a teacher has to deal with an unexpected response from a student. Black and Wiliam see moments of contingency as either synchronous where the teacher has to make a decision in real time or asynchronous such as can occur in feedback on home exercises or insights learned from a previous lesson. The first of these is particularly challenging as a teacher has to interpret a student's contribution and also make an appropriate response based on the aim of the particular lesson. Such contingent moments are, according to Corcoran (2012) "to be welcomed for the insights they give into pupils' construction of mathematical ideas" (p.136). Rowland, Thwaites, and Jared (2015) contend that a teacher's MKT is manifested in contingent moments, a point which will receive more attention later in the paper. They also argue that teachers, through reflecting on moments of contingency, have an opportunity to learn and to develop the facility to make appropriate responses to other unexpected contributions that might occur in later lessons.

In sum, meta-practices are important because they shape and are shaped by children's engagement in mathematics. For example, formative assessment should stimulate reflection on and regulation of learning; a model of a mathematical situation developed by a learner or group of learners might be used by the teacher as an object of whole-class discussion and so on. Thus, in keeping with a curricular focus on mathematization where the mathematics of the learner is prioritised, teaching is intricately linked with learning (Fosnot & Dolk, 2001). Meta-practices should permeate children's mathematical activity such as engaging in project work, solving mathematical problems in groups, using digital tools and so on. Attention is next turned to some such activities.

Practices in Integrative Contexts

As expounded by Bacon (2018), cross disciplinary integration has the potential to create meaningful, holistic learning experiences. However, in its enactment, it can lead to superficial learning with little focus on subject specific outcomes. On the other hand, the treatment of subjects as discrete and separate offers learners little opportunity to create meaningful learning links across different curricular areas. According to Russell-Bowie (2009, p.4),

> [A] balance between the two extremes is needed so that children are achieving discrete indicators and outcomes in each of the subjects and/or art forms but are also engaging in authentic learning within a meaningful, holistic context and being given the opportunity to develop generic skills as well.

She proposes an approach where broad themes or concepts are explored in a deep and meaningful way by and within different subjects. Thus the integrity of each subject is maintained whilst the possibility of learners developing generic skills such as collaboration, problem solving, or critical reasoning is enhanced. This has relevance for all curricular subjects, including STEM (an acronym for science, technology, engineering, and mathematics) which has gained considerable attention in recent years. As mentioned earlier, the prioritisation of mathematics in the recent STEMerg report is significant, particularly because internationally science is often regarded as dominating the STEM agenda (e.g., English, 2016). English (2016) argues that modeling problems can serve to ensure that mathematics has the profile it deserves within the STEM climate. As mentioned earlier, she sees mathematical modeling as stemming from complex real-world problems. In consideration of the nature of real-world problems, she cites Galbraith's (2013) four dimensions of authenticity:

> Content authenticity: the problem comprises real-world links (or links that are real for the students) and is within reach of students' mathematical knowledge.
>
> Process authenticity: the problem engages students in valid modeling processes such as posing questions, constructing a model, evaluating the model, modifying it and so on.
>
> Situation authenticity: the task requirements drive the problem-solving activity not vice versa.
>
> Product authenticity: the product developed at the end of the modeling process can be justified mathematically, and appropriately addresses the real-world problem.

As these dimensions address both generic skills and mathematical processes and outcomes, they might be used as a basis to implement Russell-Bowie's proposed approach to integration described above.

One aspect of STEM that deserves mention is the role of technology in mathematics teaching and learning. While, as described above, any tool needs to be used in a meaningful way if it is to serve the purpose of deepening mathematical understanding, technology has particular affordances (Sarama & Clements, 2016). For example, it has the potential to bring real-world applications to life in the mathematics classroom. It can support student engagement through collaborative workspaces or remote and virtual labs. It also allows for immediate feedback to students allowing them to engage in more personalised learning (e.g., Geiger, 2017; OECD, 2015). It can thus be seen that technology, used appropriately, can bring mathematics as problem-solving to the fore.

Addressing Diversity

In line with a vision of mathematics for all, an equitable curriculum is promoted in RR17 and RR18. Such a curriculum is premised on a culturally sensitive pedagogy and an acknowledgement that individuals have different ways of making sense of mathematics. It is important, however, that in taking account of culture, "white, middle-class" is not regarded as the norm or that membership of a cultural group fixes an identity on its members (Schmeichel, 2012). With regard to the latter, Sleeter (2012) argues that "[w]hat makes more sense is for teachers to bring to the classroom an awareness of diverse cultural possibilities that might relate to their students, but then to get to know the students themselves" (p.571). Warren and Miller (2016) who conducted a 4-year long numeracy project with schools in marginalised contexts in Australia propose that the following actions are included in those that make a difference to teaching mathematics:

- representing mathematical concepts in multiple ways;
- using students' engagement and assessment to inform teaching;
- implementing tailored learning experiences that cater for the participation of all learners; including those experiencing difficulties and those who need extension;
- building on students' strengths and their cultural backgrounds;
- encouraging students to orally communicate their mathematical understanding; and
- recognising that all students are capable of learning mathematics. (p.118).

These actions have been described in some of the sections above, suggesting that good pedagogy implies access by all to rich and creative mathematics.

Within-class differences in attainment in mathematics have been addressed in some primary schools by the practice of regrouping (that is, grouping or setting learners for mathematics). While this is well intentioned, it runs counter to a growth mindset, described earlier. Moreover, the practice has little effect on learning outcomes and in fact exacerbates inequities, particularly in relation to those who experience difficulty with mathematics (e.g., Boaler, Wiliam & Brown, 2000; Zevenbergen, 2005). As stated in RR17, "it is not that distinctive teaching approaches (or indeed distinctive curricula) are required but that mathematics teaching should address specific needs – including the needs of those who are exceptional because of a disability or talent, those who do not have English or Irish as a mother tongue or those coming from a disadvantaged background" (p. 124). Nonetheless, the challenge of addressing the range of needs in a classroom cannot be understated.

One framework that warrants consideration is the Universal Design for Learning (UDL) framework which, according to Hitchcock et al. (2002), derives from universal design in architecture:

> Architects have learned that designing buildings with the needs of diverse users in mind from the beginning saves costs and leads to more streamlined, accessible buildings, in which alternatives are integral to the design. And as it turns out, universal design works better for everyone. (p.9)

They maintain that if, in a similar fashion, curriculum designers recognise that learners in classrooms have diverse needs and support learning differences from the outset, the curriculum can then work for all learners. The key features of a UDL curriculum are goals, materials, methods, and assessments:

- Goals provide an appropriate challenge for all students.
- Materials have a flexible format supporting transformation between media and multiple representations of content to support all students' learning.
- Methods are flexible and diverse enough to provide appropriate learning experiences, challenges, and supports for all students.
- Assessment is sufficiently flexible to provide accurate, ongoing information that helps teachers adjust instruction and maximize learning. (Hitchcock, Meyer, Rose, & Jackson, 2002, p.9)

Hunt & Andreasen (2011) give an example of the application of UDL principles to a mathematics lesson in middle school (7th grade). A curriculum designed on these principles makes considerable demands of teachers and thus high-quality professional development, the subject of the next section, is critical.

Mathematics Teacher Development

The redeveloped mathematics curriculum represents a transformation in how mathematics itself, the learning of mathematics and concomitant pedagogy are conceived. As described by Warren & Miller (2016), amongst others, a robust and sustained professional development programme should be provided to allow substantive teacher change to occur. Llinares and Krainer (2006) suggest that pre-service and in-service teachers develop MKT by

> a. actively working on (mathematical, educational) problems;
>
> b. critically reflecting on practice (solving mathematical problems, observing other teachers' lessons, analysing own teaching); and
>
> c. sharing knowledge, networking, and participating in different kinds of communities.

Particular features of CPD are teacher autonomy and the incorporation of action-research in (a), (b) and (c) above. A rich site for these activities is Lesson Study (LS), which is given prominence in pre-service teacher mathematics education in Ireland (Corcoran, 2011; Leavy & Hourigan, 2016; Ní Shúilleabháin, 2016). As proposed in RR18, it should be given due attention in the implementation of the redeveloped curriculum. It was also suggested that a framework such as the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) provides a language to describe mathematical teaching activity – including the aforementioned contingency moments – and can facilitate a growth in MKT. Recently some attention has been given to ways in which LS and the Teaching for Robust Understanding (TRU) framework can be interwoven (Schoenfeld, Dosalmas, Fink, et al., 2019). The TRU framework was developed to describe equitable and robust learning environments (Schoenfeld and the Teaching for Robust Understanding Project, 2016). At its core are five dimensions of classroom activity – content (e.g., mathematics); cognitive demand; equitable access to content; agency authority and identity; and formative assessment. Schoenfeld and the project team maintain that a focus on these five dimensions in a mathematics classroom underpins powerful mathematical thinking in learners. The TRU framework and LS both focus on learners' experience of mathematics and, according to Schoenfeld et al. (2019), the TRU framework can support each stage of the LS cycle, that is studying curriculum and formulating goals; planning, conducting research lesson; and reflecting (Lewis, 2002). The TRU framework and meta-practices described above also find common ground and therefore could be considered in the redevelopment of the primary mathematics curriculum.

Conclusion

This paper focuses on important issues in mathematics learning and teaching for children in middle/senior primary classes. It builds on the 2014 research reports (RR17 and RR18) that are the bases for a redeveloped mathematics curriculum for children aged 3-8 years. Consideration is given to findings on the performance of primary children in recent national and international tests of mathematical achievement, and to current curricular developments in the primary and post-primary sectors. Some powerful mathematical ideas - conceptualised as those that transcend content domains - for this age group are explored. Problem-solving situations that have potential to develop children mathematical thinking are discussed. The meta-practices, which should permeate children's engagement with mathematics, are elaborated. The place of mathematics in STEM is given attention, as are equity issues and mathematics teacher development.

In order to bring about the change in the mathematics curriculum, a different conceptualisation of what mathematics is and what it means to do mathematics is required at school and societal level. Traditionally mathematics has been regarded as an objective body of facts to be transmitted from one generation to the next. Vestiges of this view still remain and can be seen when there is an over emphasis on procedures in mathematics teaching and learning, or in the perception that "being good at mathematics" is the preserve of a few. Current views of mathematics – espoused here and in RR17 and RR18 – rest on ideas of human and cultural activity. From this perspective, all individuals have an innate ability to make sense of the world through mathematics. Furthermore, discursive activity is viewed as intrinsic to and not separate from mathematics. Similarly, problem solving is not a route to mathematical objects but rather deeply embedded in what it means to "do maths". An important implication is that learning paths where mathematical content and processes are interwoven are presented in curriculum documentation.

The pedagogical approaches espoused in this paper put learners at the heart of the classroom. A range of research is cited to show that they develop mathematical thinking by engaging in problems that are of interest to them, by communicating their thinking, by experiencing cognitive challenge, by modelling their sense-making, and by receiving appropriate intervention when necessary. It is important to note that these approaches – albeit with a slight change of emphasis – apply across the continuum of education from early years to post-primary and beyond. Teachers play a central role in creating the kind of learning environment described above. The significance of MKT has been referenced several times in this paper. With its emphasis on children's thinking, collaborative practice and analysis of teaching, Lesson Study is promoted here and elsewhere as a highly effective means of developing this knowledge. It should be integral to the mathematics professional development but needs systemic support to allow teachers time and space to engage in it. Moreover, curriculum materials should exemplify effective mathematics teaching, engagement by children in mathematics, and instances of meta-practices in action.

Perspectives on mathematics as "human" and "cultural" have implications not only for greater equity in access to mathematics but also for the future needs of learners since the ability to collaborate and problem solve is required for an increasingly complex society. It is incumbent on those involved in curriculum development – including teachers and schools – to ensure that the mathematics at its centre are important with regard to future needs of learners. This means that the powerful mathematical ideas described earlier in this paper, together with those that emerge in time, should be interrogated for teaching, learning and assessment purposes. In order to support this, research into children's learning paths within and across different mathematical domains is warranted. The findings from such research, as they become available, should be incorporated into the curriculum. This calls for a redeveloped mathematics curriculum that is dynamic, flexible and, above all, ambitious in ensuring that all children have access to deep and challenging mathematical ideas.

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Appendix

Features of Good Mathematics Pedagogy[2]

People and Relationships
Strong interpersonal relationships within the setting are fundamental to children's progress.
The classroom atmosphere is one in which all children are comfortable with making contributions.
The diverse cultures of children and their families are taken seriously and treated as classroom resources.
Co-construction of mathematical knowledge is developed through the respectful discussion and exchange of ideas.
The Learning Environment
The starting point for teaching is children's current knowledge and interests.
Classroom activity and discourse focus explicitly on mathematical ideas and problems.
Tasks are designed based on children's current interests, but they also serve the long-term learning goals.
Children are given opportunities to engage in justification, argumentation and generalisation. In this way they learn to use the language of mathematics.
A wide range of children's everyday activities, play and interests are used to engage, challenge and extend their mathematical knowledge and skills.
Learning environments that are rich in the use of a wide range of tools (including digital tools) that support all children's mathematical learning.
Children are provided with opportunities to learn in a wide range of imaginative and real-world contexts, some of which integrate and connect mathematics with other activities and other activities with mathematics.
Investigative-type activities that stem from children's interests and questions, give rise to the creation of models of the problem which can be generalised and used in other situations.
Contexts that are rich in perceptual and social experiences are used to support the development of problem-solving and creative skills.
Children experience opportunities to learn through creative and playful approaches where they generate ideas around the solution of problems that are of interest to them.
The potential of everyday activities is recognised and harnessed.
Opportunities are balanced for children to learn in small groups, in the whole-class group and individually, as appropriate.
Teaching is based on appropriate sequencing. Whilst learning paths are used to provide a general overview of the learning continua of the group of children, this is tempered with the knowledge that children do not all progress along a common developmental path.
Planned and spontaneous learning opportunities are used to promote mathematics learning.
The Learner
Children's reasoning is at the centre of instructional decision-making and planning.
Teaching is continually adjusted according to children's learning and as a result of on-going assessment.
Scaffolding that extends children's mathematical thinking is provided while children's contributions are simultaneously valued.
Opportunities are provided for children to engage in metacognitive-like activities such as planning and reflecting. In doing so, children are supported to set their own goals and assess their own achievements.
Assessment is carried out in the context of adult-child interactions and involves some element of sustained, shared thinking.

[2] Original source of table is Dooley et al. (2014). Mathematics in Early Childhood and Primary Education (3-8 years) Teaching and Learning. Research Report No. 18. NCCA. Dublin